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# A DE MONTESSUS TYPE CONVERGENCE STUDY FOR A VECTOR-VALUED RATIONAL INTERPOLATION PROCEDURE

BY

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ABSTRACT

In a recent paper of the author [8], three new interpolation procedures for vector-valued functions $F(z)$, where $F : \mathbb{C} \to \mathbb{C}^N$, were proposed, and some of their algebraic properties were studied. In the present work, we concentrate on one of these procedures, denoted IMMPE, and study its convergence properties when it is applied to meromorphic functions. We prove de Montessus and Koenig type theorems in the presence of simple poles when the points of interpolation are chosen appropriately. We also provide simple closed-form expressions for the error in case the function $F(z)$ in question is itself a vector-valued rational function whose denominator polynomial has degree greater than that of the interpolant.

## 1. Introduction

In a recent work, Sidi [8], we presented three different kinds of vector-valued rational interpolation procedures. These were modelled after some rational approximation procedures from Maclaurin series of vector-valued functions developed in Sidi [6], which in turn had their origin in vector extrapolation methods. Vector extrapolation methods are used for accelerating the convergence of certain kinds of vector sequences, such as those produced by fixed-point iterative methods on linear and nonlinear systems of algebraic equations.

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Some of the algebraic properties of these interpolants have already been mentioned in [8]. The study of algebraic properties is continued in another paper [9] by the author. In the present work, we continue to study one of the three interpolation procedures that was denoted IMMPE in [8]. In particular, we concentrate on the convergence properties of IMMPE as it is being applied to vector-valued meromorphic functions.

In the next section, we provide a brief description of the interpolants that result from IMMPE. Following this, in Section 3, we derive a closed-form expression for the error when the function $F(z)$ being interpolated is rational with simple poles. The main results of this section are Theorems 3.6 and 3.8. In Section 4, we present the assumption we make about the points of interpolation and its consequences.

Starting with the developments of Section 3, in Sections 5 and 6, we present a detailed convergence theory, concerning meromorphic vector-valued functions $F(z)$ with simple poles, for sequences of interpolants whose denominators are of a fixed degree, while the number of interpolation conditions (i.e., the degree of the numerators) tends to infinity. This theory provides us with de Montessus and Koenig type theorems for $R_{p,k}(z)$ as $p \to \infty$, while $k$ is held fixed. Section 5 concerns vector-valued rational functions; its main results are Theorems 5.1 and 5.2, which concern the denominators of the IMMPE approximants $R_{p,k}(z)$, and Theorem 5.3, which concerns the convergence of $R_{p,k}(z)$, all as $p \to \infty$. Section 6 concerns general vector-valued meromorphic functions, and its results are obtained by extending those of Section 5; Theorems 6.2, 6.3, and 6.4, the main results of this section, are extensions of Theorems 5.1, 5.2, and 5.3, respectively.

Our theory is in the spirit of that given by Saff [4] for the scalar rational interpolation problem and by Graves-Morris and Saff [2] for vector-valued rational interpolants, while the technique used here is analogous to that developed by Sidi, Ford, and Smith [10] and used by Sidi [5] in the study of Padé approximants, hence different from that of [4] and [2] . In addition, the technique we use here enables us to obtain optimally refined results in the form of asymptotic equalities.

## 2. Definition and algebraic properties of IMMPE

To set the stage for later developments, and to fix the notation as well, we start with a brief description of the developments in [8] and [9] that concern IMMPE.

Let $z$ be a complex variable and let $F(z)$ be a vector-valued function such that $F : \mathbb{C} \to \mathbb{C}^N$. Assume that $F(z)$ is defined on a bounded open set $\Omega \subset \mathbb{C}$ and consider the problem of interpolating $F(z)$ at some of the points $\xi_1, \xi_2, \ldots,$ in this set. We do not assume that the $\xi_i$ are necessarily distinct. The general picture is described in the next paragraph.

Let $a_1, a_2, \ldots,$ be distinct complex numbers, and let

$$\begin{aligned} &\xi_1 = \xi_2 = \cdots = \xi_{r_1} = a_1 \\ &\xi_{r_1+1} = \xi_{r_1+2} = \cdots = \xi_{r_1+r_2} = a_2 \\ &\xi_{r_1+r_2+1} = \xi_{r_1+r_2+2} = \cdots = \xi_{r_1+r_2+r_3} = a_3 \\ &\text{and so on.} \end{aligned} \tag{2.1}$$

Let $G_{m,n}(z)$ be the vector-valued polynomial (of degree at most $n-m$) that interpolates $F(z)$ at the points $\xi_m, \xi_{m+1}, \ldots, \xi_n$ in the generalized Hermite sense. Thus, in Newtonian form, this polynomial is given as in (see, e.g., Stoer and Bulirsch [11, Chapter 2] or Atkinson [1, Chapter 3])

$$\begin{aligned} G_{m,n}(z) =& F[\xi_m] + F[\xi_m, \xi_{m+1}](z-\xi_m) \\ &+ F[\xi_m, \xi_{m+1}, \xi_{m+2}](z-\xi_m)(z-\xi_{m+1}) + \cdots \\ &+ F[\xi_m, \xi_{m+1}, \ldots, \xi_n](z-\xi_m)(z-\xi_{m+1})\cdots(z-\xi_{n-1}). \end{aligned} \tag{2.2}$$

Here, $F[\xi_r, \xi_{r+1}, \ldots, \xi_{r+s}]$ is the divided difference of order $s$ of $F(z)$ over the set of points $\{\xi_r, \xi_{r+1}, \ldots, \xi_{r+s}\}$. Obviously, $F[\xi_r, \xi_{r+1}, \ldots, \xi_{r+s}]$ are all vectors in $\mathbb{C}^N$. Also, $F[\xi_m] = F(\xi_m)$.

We define the scalar polynomials $\psi_{m,n}(z)$ via

$$\psi_{m,n}(z) = \prod_{r=m}^{n}(z-\xi_r), \quad n \geq m \geq 1; \quad \psi_{m,m-1}(z) = 1, \quad m \geq 1. \tag{2.3}$$

We also define the vectors $D_{m,n}$ via

$$D_{m,n} = F[\xi_m, \xi_{m+1}, \ldots, \xi_n], \quad n \geq m. \tag{2.4}$$

With this notation, we can rewrite (2.2) in the form

$$G_{m,n}(z) = \sum_{i=m}^{n} D_{m,i}\, \psi_{m,i-1}(z). \tag{2.5}$$

The vector-valued rational interpolants to the function $F(z)$ we developed in [8] are all of the general form

$$R_{p,k}(z) = \frac{U_{p,k}(z)}{V_{p,k}(z)} = \frac{\sum_{j=0}^{k} c_j\, \psi_{1,j}(z)\, G_{j+1,p}(z)}{\sum_{j=0}^{k} c_j\, \psi_{1,j}(z)}, \tag{2.6}$$

where $c_0, c_1, \ldots, c_k$ are, for the time being, arbitrary complex scalars, and $p$ is an arbitrary integer. Obviously, $U_{p,k}(z)$ is a vector-valued polynomial of degree at most $p-1$ and $V_{p,k}(z)$ is a scalar polynomial of degree at most $k$. It is also clear from (2.6) that $k \leq p-1$.

The following theorem says that, whether the $\xi_i$ are distinct or not, $R_{p,k}(z)$ interpolates $F(z)$. See [8, Lemmas 2.1 and 2.3].

THEOREM 2.1: *Let the vector-valued rational function $R_{p,k}(z)$ be as in (2.6), and assume that $V_{p,k}(\xi_i) \neq 0$, $i = 1, \ldots, p$.*

(i) *When the $\xi_i$ are distinct, $R_{p,k}(z)$ interpolates $F(z)$ at the points $\xi_1, \xi_2, \ldots, \xi_p$ in the ordinary sense:*

$$R_{p,k}(\xi_i) = F(\xi_i), \quad i = 1, \ldots, p. \tag{2.7}$$

(ii) *When the $\xi_i$ are not necessarily distinct and are ordered as in (2.1), $R_{p,k}(z)$ interpolates $F(z)$ in the generalized Hermite sense as follows: Let $t$ and $\rho$ be the unique integers satisfying $t \geq 0$ and $0 \leq \rho < r_{t+1}$ for which $p = \sum_{i=1}^{t} r_i + \rho$. Then,*

$$\begin{aligned} R_{p,k}^{(s)}(a_i) = F^{(s)}(a_i), \quad &\text{for } \; s = 0, 1, \ldots, r_i - 1 \;\text{ when }\; i = 1, \ldots, t, \\ &\text{and for } \; s = 0, 1, \ldots, \rho - 1 \;\text{ when }\; i = t+1. \end{aligned} \tag{2.8}$$

*Of course, when $\rho = 0$, there is no interpolation at $a_{t+1}$.*

*Remark:* It must be noted that the condition $V_{p,k}(\xi_i) \neq 0$, $i = 1, \ldots, p$, features throughout this work. Because $k < p$ and because $p$ can be arbitrarily large, this condition might look too restrictive at first. This is not the case, however. Indeed, the condition $V_{p,k}(\xi_i) \neq 0$, $i = 1, \ldots, p$, is natural for the following reason: Normally, we take the points of interpolation $\xi_i$ in a set $\Omega$ on which the function $F(z)$ is regular. If $R_{p,k}(z)$ is to approximate $F(z)$, it should also be a regular function over $\Omega$ and hence free of singularities there. Since the singularities of $R_{p,k}(z)$ are the zeros of $V_{p,k}(z)$, this implies that $V_{p,k}(z)$ should not vanish on $\Omega$. (We expect the singularities of $R_{p,k}(z)$ —the zeros of $V_{p,k}(z)$— to be close to the singularities of $F(z)$, which are outside the set $\Omega$.)

So far, the $c_j$ in (2.6) are arbitrary. Of course, the quality of $R_{p,k}(z)$ as an approximation to $F(z)$ depends very strongly on the choice of the $c_j$. Naturally, the $c_j$ must depend on $F(z)$ and on the $\xi_i$. Fixing the integers $k$ and $p$ such that $p \geq k+1$, we define the $c_j$ for IMMPE to be the solution to the system of equations

$$\left(q_i, \sum_{j=0}^{k} c_j D_{j+1,p+1}\right) = 0, \quad i = 1, \dots, k; \quad c_k = 1, \tag{2.9}$$

where $q_1, \dots, q_k$ are linearly independent constant vectors in $\mathbb{C}^N$. Note that these equations form the linear system

$$\sum_{j=0}^{k} u_{i,j} c_j = -u_{i,k}, \quad i = 1, \dots, k; \quad c_k = 1; \quad u_{i,j} = \big(q_i, D_{j+1,p+1}\big). \tag{2.10}$$

It has been shown in [8] that, provided a unique solution to these equations exists, $R_{p,k}(z)$ has a determinantal representation given as in

$$R_{p,k}(z) = \frac{P(z)}{Q(z)} = \frac{\begin{vmatrix} \psi_{1,0}(z)\, G_{1,p}(z) & \psi_{1,1}(z)\, G_{2,p}(z) & \cdots & \psi_{1,k}(z)\, G_{k+1,p}(z) \\ u_{1,0} & u_{1,1} & \cdots & u_{1,k} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k} \end{vmatrix}}{\begin{vmatrix} \psi_{1,0}(z) & \psi_{1,1}(z) & \cdots & \psi_{1,k}(z) \\ u_{1,0} & u_{1,1} & \cdots & u_{1,k} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k} \end{vmatrix}}. \tag{2.11}$$

Here, the numerator determinant $P(z)$ is vector-valued and is defined by its expansion with respect to its first row. That is, if $M_j$ is the cofactor of the term $\psi_{1,j}(z)$ in the denominator determinant $Q(z)$, then

$$R_{p,k}(z) = \frac{\sum_{j=0}^{k} M_j\, \psi_{1,j}(z)\, G_{j+1,p}(z)}{\sum_{j=0}^{k} M_j\, \psi_{1,j}(z)}. \tag{2.12}$$

Note that this determinantal representation has been used throughout [9] extensively. It seems to offer a very effective tool for the study of $R_{p,k}(z)$, as we will see later in this work as well.

Here is a summary of the results of [9]:

(1) A sufficient condition for the equations in (2.9) to have a unique solution is that (see [9, Lemma 2.1 and Theorem 2.2])

$$\begin{vmatrix} u_{1,0} & u_{1,1} & \cdots & u_{1,k-1} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k-1} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k-1} \end{vmatrix} \neq 0; \quad u_{i,j} = \big(q_i, D_{j+1,p+1}\big). \tag{2.13}$$

This also guarantees the uniqueness of $R_{p,k}(z)$ provided $V_{p,k}(\xi_i) \neq 0$, $i = 1, \ldots, p$. For (2.13) to be true, it is necessary (but not sufficient) that the vectors $D_{1,p+1}, D_{2,p+1}, \ldots, D_{k,p+1}$ be linearly independent, just as $q_1, q_2, \ldots, q_k$ are. It is shown in [9, Sections 2 and 5] that this holds when $F(z)$ is a vector-valued rational function of the form

$$F(z) = u(z) + \sum_{s=1}^{\sigma} \sum_{j=1}^{r_s} \frac{v_{sj}}{(z - z_s)^j}, \tag{2.14}$$

where $u(z)$ is an arbitrary vector-valued polynomial, the vectors $v_{sj} \in \mathbb{C}^N$, $1 \leq j \leq r_s$, $1 \leq s \leq \sigma$, are linearly independent, $z_1, \ldots, z_\sigma$ are distinct points in $\mathbb{C}$, and $k \leq \sum_{s=1}^{\sigma} r_s \leq N$.

(2) The denominator polynomial $V_{p,k}(z)$ of the IMMPE interpolant $R_{p,k}(z)$ is a symmetric function of all the $\xi_i$ used to construct it, namely, of $\xi_1, \xi_2, \ldots, \xi_{p+1}$, while $R_{p,k}(z)$ itself is a symmetric function of the points of interpolation, namely, of $\xi_1, \xi_2, \ldots, \xi_p$. That is, $R_{p,k}(z)$ is independent of the order of the interpolation points $\xi_1, \ldots, \xi_p$. See [9, Lemma 3.4 and Theorem 3.5].

(3) Under certain conditions, IMMPE produces $F(z)$ when the latter is a vector-valued rational function. Specifically, when $F(z) = \tilde{U}(z)/\tilde{V}(z)$, where $\tilde{U}(z)$ is a vector-valued polynomial of degree at most $p-1$ and $\tilde{V}(z)$ is scalar a polynomial of degree exactly $k$, there holds $R_{p,k}(z) \equiv F(z)$ provided (2.13) and $V_{p,k}(\xi_i) \neq 0$, $i = 1, \ldots, p$, hold. See [9, Theorem 4.1].

## 3. IMMPE error formula when $F(z)$ is a vector-valued rational function

We start our study of IMMPE for the case in which the function $F(z)$ is a vector-valued rational function with simple poles, namely,

$$F(z) = \sum_{s=1}^{\mu} \frac{v_s}{z - z_s} + u(z), \tag{3.1}$$

where $u(z)$ is an arbitrary vector-valued polynomial, $z_1, \ldots, z_\mu$ are distinct points in the complex plane, and $v_1, \ldots, v_\mu$ are linearly independent constant vectors in $\mathbb{C}^N$. Clearly, $\mu \leq N$.

*Example:* Let $A$ be an $N \times N$ matrix and $b$ an $N$-vector, and consider the solution to the linear system of equations $(I - zA)x = b$. This solution is $x = F(z) \equiv (I - zA)^{-1}b$ and precisely of the form described in (3.1) with $z_s \neq 0, s = 1, \ldots, \mu$, provided the nonzero eigenvalues of $A$ are nondefective, that is, they have only corresponding eigenvectors but no principal vectors. In case $A$ is also nonsingular (hence diagonalizable as well), there holds $u(z) \equiv 0$ and $\mu \leq N$ in (3.1); otherwise, $\mu < N$ and $\deg(u) + \mu \leq N - 1$. In addition, $Av_s = v_s/z_s,\ s = 1, \ldots, \mu$, in (3.1). To see this, it is sufficient to look at the following special case.

Denote the eigenvalues of $A$ by $\lambda_1, \ldots, \lambda_N$, and assume that $\lambda_i \neq 0$, $i = 1, \ldots, M$, are simple and distinct so that there is precisely one eigenvector $v_i$ corresponding to $\lambda_i$ and, in case $M < N$, $\lambda_i = 0$, $i = M + 1, \ldots, N$, and that there is only one Jordan block with zero eigenvalue. This means that there exists an $N \times N$ nonsingular matrix $P$,

$$P = \big[v_1 \,|\, v_2 \,|\, \cdots \,|\, v_M \,|\, w_1 \,|\, w_2 \,|\, \cdots \,|w_{M_0}\big], \quad M_0 = N - M,$$

and that

$$Av_i = \lambda_i v_i, \quad i = 1, \ldots, M; \quad Aw_1 = 0, \quad Aw_i = w_{i-1}, \quad i = 2, \ldots, M_0.$$

Now, $b = \sum_{i=1}^{M} \alpha_i v_i + \sum_{i=1}^{M_0} \beta_i w_i$, for some scalars $\alpha_i$ and $\beta_i$. Next, let us expand $x$ in terms of the $v_i$ and $w_i$ in the form $x = \sum_{i=1}^{M} \gamma_i v_i + \sum_{i=1}^{M_0} \delta_i w_i$. Substituting these expansions in $(I - zA)x = b$, and equating the coefficients of the $v_i$ and $w_i$ on both sides, we obtain

$$(1 - z\lambda_i)\gamma_i = \alpha_i, \quad 1 \leq i \leq M; \quad \delta_{M_0} = \beta_{M_0}, \quad \delta_i = \beta_i + z\delta_{i+1}, \quad 1 \leq i \leq M_0 - 1.$$

Solving these equations, we see that $\gamma_i = \alpha_i/(1 - z\lambda_i)$, $1 \le i \le M$, and that $\delta_i$ is a polynomial in $z$ of degree at most $M_0 - i$. We also see that $z_s = 1/\lambda_s$ in (3.1), and that $\deg(u) \le M_0 - 1$. When $M = N$, hence all eigenvalues are nonzero, we have $u(z) \equiv 0$.

The truth of the assertion that $F(z) = (I - zA)^{-1}b$ is as in (3.1) also in the general case can now be shown in exactly the same way.

We now develop some technical tools that we will use throughout this work. The next lemma was stated and proved as Lemma A.1 in [10].

Lemma 3.1: *Let $i_0, i_1, \ldots, i_k$ be positive integers, and assume that the scalars $v_{i_0,i_1,\ldots,i_k}$ are odd under an interchange of any two of the indices $i_0, i_1, \ldots, i_k$. Let $t_{i,j}$, $i, j \ge 1$, be scalars and let $\sigma_i$, $i \ge 1$ be all scalars or vectors. Define*

$$I_{k,N} = \sum_{i_0=1}^{N} \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} \sigma_{i_0} \left( \prod_{p=1}^{k} t_{i_p,p} \right) v_{i_0,i_1,\ldots,i_k}$$

*and*

$$J_{k,N} = \sum_{1 \le i_0 < i_1 < \cdots < i_k \le N} \begin{vmatrix} \sigma_{i_0} & \sigma_{i_1} & \cdots & \sigma_{i_k} \\ t_{i_1,1} & t_{i_2,1} & \cdots & t_{i_k,1} \\ t_{i_1,2} & t_{i_2,2} & \cdots & t_{i_k,2} \\ \vdots & \vdots & & \vdots \\ t_{i_1,k} & t_{i_2,k} & \cdots & t_{i_k,k} \end{vmatrix} v_{i_0 i_1,\ldots,i_k} \, .$$

*Then*

$$I_{k,N} = J_{k,N}.$$

The next lemma is Lemma 1.2 in [7].

Lemma 3.2: *Let $Q_i(x) = \sum_{j=0}^{i} a_{ij} x^j$, with $a_{ii} \ne 0$, $i = 0, 1, \ldots, n$, and let $x_i$, $i = 0, 1, \ldots, n$, be arbitrary complex numbers. Then*

$$\begin{vmatrix} Q_0(x_0) & Q_0(x_1) & \cdots & Q_0(x_n) \\ Q_1(x_0) & Q_1(x_1) & \cdots & Q_1(x_n) \\ \vdots & \vdots & & \vdots \\ Q_n(x_0) & Q_n(x_1) & \cdots & Q_n(x_n) \end{vmatrix} = \left( \prod_{i=0}^{n} a_{ii} \right) V(x_0, x_1, \ldots, x_n), \tag{3.2}$$

*where $V(x_0, x_1, \ldots, x_n) = \prod_{0 \le i < j \le n} (x_j - x_i)$ is a Vandermonde determinant.*

Lemma 3.3: *Let $\omega_a(z) = (z-a)^{-1}$. Then, $\omega_a[\xi_m, \dots, \xi_n]$, the divided difference of $\omega_a(z)$ over the set of points $\{\xi_m, \dots, \xi_n\}$, is given by*

$$\omega_a[\xi_m, \dots, \xi_n] = -\frac{1}{\psi_{m,n}(a)} = -\frac{\psi_{1,m-1}(a)}{\psi_{1,n}(a)}. \tag{3.3}$$

*This is true whether the $\xi_i$ are distinct or not.*

*Proof.* When the $\xi_i$ are distinct, the truth of the assertion in (3.3) can be shown by induction using the recursion relation satisfied by divided differences, namely,

$$H[\xi_r, \xi_{r+1}, \dots, \xi_{r+s}] = \frac{H[\xi_r, \xi_{r+1}, \dots, \xi_{r+s-1}] - H[\xi_{r+1}, \xi_{r+2}, \dots, \xi_{r+s}]}{\xi_r - \xi_{r+s}}, \quad r = 1, 2, \dots, \quad s = 1, 2, \dots, \tag{3.4}$$

with the initial conditions

$$H[\xi_r] = H(\xi_r), \quad r = 1, 2, \dots \, . \tag{3.5}$$

In case the $\xi_i$ are not distinct, we invoke the fact that, if $H(z)$ is continuously differentiable $n-m+1$ times in a domain containing the set $\{\xi_m, \dots, \xi_n\}$, then $H[\xi_m, \dots, \xi_n]$ is a continuous function of these $\xi_i$. ∎

Lemma 3.4: *Let $F(z)$ be given as in* (3.1). *Let $n - m > \deg(u)$. Then, the following are true:*

(i) $D_{m,n} = F[\xi_m, \dots, \xi_n]$ *is given as in*

$$D_{m,n} = -\sum_{s=1}^{\mu} \frac{v_s}{\psi_{m,n}(z_s)} = -\sum_{s=1}^{\mu} v_s \frac{\psi_{1,m-1}(z_s)}{\psi_{1,n}(z_s)}. \tag{3.6}$$

*Therefore, we also have*

$$\big(q_i, D_{m,n}\big) = -\sum_{s=1}^{\mu} \frac{\alpha_{i,s}}{\psi_{m,n}(z_s)} = -\sum_{s=1}^{\mu} \alpha_{i,s} \frac{\psi_{1,m-1}(z_s)}{\psi_{1,n}(z_s)}, \quad \alpha_{i,s} = \big(q_i, v_s\big). \tag{3.7}$$

(ii) $F(z) - G_{m,n}(z) = \psi_{m,n}(z) F[z, \xi_m, \dots, \xi_n]$ *is given as in*

$$F(z) - G_{m,n}(z) = \psi_{m,n}(z) \sum_{s=1}^{\mu} e_s(z) \frac{\psi_{1,m-1}(z_s)}{\psi_{1,n}(z_s)}; \quad e_s(z) = \frac{v_s}{z - z_s}. \tag{3.8}$$

*This is true whether the $\xi_i$ are distinct or not.*

*Proof.* The result follows from Lemma 3.3 and from the fact that $u[\xi_m, \dots, \xi_n] = 0$ and $u[z, \xi_m, \dots, \xi_n] = 0$ because $u(z)$ is a polynomial and $n - m > \deg(u)$. ∎

The next lemma, whose proof we leave to the reader, gives the determinant representation of $F(z) - R_{p,k}(z)$, and we will be analyzing it in the sequel.

Lemma 3.5: *Let*

$$\Delta_j(z) = \psi_{1,j}(z)\big[F(z) - G_{j+1,p}(z)\big], \quad j = 0, 1, \ldots . \tag{3.9}$$

*Then the error in $R_{p,k}(z)$ has the determinantal representation*

$$F(z) - R_{p,k}(z) = \Delta(z)/Q(z), \tag{3.10}$$

*where $Q(z)$ is the denominator determinant in* (2.11) *and*

$$\Delta(z) = \begin{vmatrix} \Delta_0(z) & \Delta_1(z) & \cdots & \Delta_k(z) \\ u_{1,0} & u_{1,1} & \cdots & u_{1,k} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k} \end{vmatrix}. \tag{3.11}$$

3.1. Algebraic structure of $Q(z)$. We start with the analysis of $Q(z)$, the denominator determinant of $R_{p,k}(z)$ in (2.11). The following theorem gives a closed form expression for $Q(z)$ in simple terms.

Theorem 3.6: *Let $F(z)$ be the vector-valued rational function in* (3.1), *and precisely as described in the first paragraph of this section, with the notation therein. With $\alpha_{i,s}$ as in* (3.7), *define*

$$T_{s_1,\ldots,s_k} = \begin{vmatrix} \alpha_{1,s_1} & \alpha_{1,s_2} & \cdots & \alpha_{1,s_k} \\ \alpha_{2,s_1} & \alpha_{2,s_2} & \cdots & \alpha_{2,s_k} \\ \vdots & \vdots & & \vdots \\ \alpha_{k,s_1} & \alpha_{k,s_2} & \cdots & \alpha_{k,s_k} \end{vmatrix}. \tag{3.12}$$

*Let also*

$$\Psi_p(z) = \psi_{1,p+1}(z). \tag{3.13}$$

*Then, with $p > k + \deg(u)$,*

(3.14)

$$Q(z) = (-1)^k \sum_{1 \le s_1 < s_2 < \cdots < s_k \le \mu} T_{s_1,\ldots,s_k} V(z, z_{s_1}, z_{s_2}, \ldots, z_{s_k}) \left[\prod_{i=1}^{k} \Psi_p(z_{s_i})\right]^{-1}.$$

*Proof.* Taking $p > k + \deg(u)$, and invoking Lemma 3.4 in the determinant $Q(z)$, we first have

$$u_{i,j} = \big(q_i, D_{j+1,p+1}\big) = -\sum_{s=1}^{\mu} \alpha_{i,s} \frac{\psi_{1,j}(z_s)}{\Psi_p(z_s)}. \tag{3.15}$$

Thus,

$$Q(z) = (-1)^k \begin{vmatrix} \psi_{1,0}(z) & \psi_{1,1}(z) & \cdots & \psi_{1,k}(z) \\ \sum_{s_1} \alpha_{1,s_1} \frac{\psi_{1,0}(z_{s_1})}{\Psi_p(z_{s_1})} & \sum_{s_1} \alpha_{1,s_1} \frac{\psi_{1,1}(z_{s_1})}{\Psi_p(z_{s_1})} & \cdots & \sum_{s_1} \alpha_{1,s_1} \frac{\psi_{1,k}(z_{s_1})}{\Psi_p(z_{s_1})} \\ \sum_{s_2} \alpha_{2,s_2} \frac{\psi_{1,0}(z_{s_2})}{\Psi_p(z_{s_2})} & \sum_{s_2} \alpha_{2,s_2} \frac{\psi_{1,1}(z_{s_2})}{\Psi_p(z_{s_2})} & \cdots & \sum_{s_2} \alpha_{2,s_2} \frac{\psi_{1,k}(z_{s_2})}{\Psi_p(z_{s_2})} \\ \vdots & \vdots & & \vdots \\ \sum_{s_k} \alpha_{k,s_k} \frac{\psi_{1,0}(z_{s_k})}{\Psi_p(z_{s_k})} & \sum_{s_k} \alpha_{k,s_k} \frac{\psi_{1,1}(z_{s_k})}{\Psi_p(z_{s_k})} & \cdots & \sum_{s_k} \alpha_{k,s_k} \frac{\psi_{1,k}(z_{s_k})}{\Psi_p(z_{s_k})} \end{vmatrix}.$$

Because determinants are multilinear in their rows (and columns), we can take the summations outside. Following that, we take out the common factors from each row of the remaining determinant. We obtain

$$Q(z) = (-1)^k \sum_{s_1} \sum_{s_2} \cdots \sum_{s_k} \left( \prod_{i=1}^{k} \alpha_{i,s_i} \right) \left[ \prod_{i=1}^{k} \Psi_p(z_{s_i}) \right]^{-1} X(z, z_{s_1}, z_{s_2}, \ldots, z_{s_k}),$$

where

$$X(y_0, y_1, y_2, \ldots, y_n) = \begin{vmatrix} \psi_{1,0}(y_0) & \psi_{1,1}(y_0) & \cdots & \psi_{1,k}(y_0) \\ \psi_{1,0}(y_1) & \psi_{1,1}(y_1) & \cdots & \psi_{1,k}(y_1) \\ \vdots & \vdots & & \vdots \\ \psi_{1,0}(y_k) & \psi_{1,1}(y_k) & \cdots & \psi_{1,k}(y_k) \end{vmatrix}. \tag{3.16}$$

Now, since $\psi_{1,r}(z)$ is a monic polynomial in $z$ of degree $r$, Lemma 3.2 applies, and we also have

$$X(y_0, y_1, \ldots, y_n) = V(y_0, y_1, \ldots, y_n) = \prod_{0 \le i < j \le n} (y_j - y_i), \tag{3.17}$$

is the Vandermonde determinant. Since the product

$$\left[ \prod_{i=1}^{k} \Psi_p(z_{s_i}) \right]^{-1} X(z, z_{s_1}, z_{s_2}, \ldots, z_{s_k})$$

is odd under an interchange of any two of the indices $s_1, \ldots, s_k$, Lemma 3.1 applies, and we obtain the result in (3.14). ∎

3.2. Algebraic structure of $\Delta(z)$. We now turn to $\Delta(z)$, the numerator determinant of $F(z) - R_{p,k}(z)$ in Lemma 3.5.

Theorem 3.7: *Let $F(z)$ be the vector-valued rational function in* (3.1), *and precisely as described in the first paragraph of this section, with the notation therein. With $\alpha_{i,s}$ and $e_s(z)$ as in* (3.7) *and* (3.8), *respectively, define*

$$\widehat{e}_s^{(p)}(z) = e_s(z)(z_s - \xi_{p+1}) \tag{3.18}$$

*and*

$$\widehat{T}^{(p)}_{s_0,s_1,\ldots,s_k}(z) = \begin{vmatrix} \widehat{e}^{(p)}_{s_0}(z) & \widehat{e}^{(p)}_{s_1}(z) & \cdots & \widehat{e}^{(p)}_{s_k}(z) \\ \alpha_{1,s_0} & \alpha_{1,s_1} & \cdots & \alpha_{1,s_k} \\ \alpha_{2,s_0} & \alpha_{2,s_1} & \cdots & \alpha_{2,s_k} \\ \vdots & \vdots & & \vdots \\ \alpha_{k,s_0} & \alpha_{k,s_1} & \cdots & \alpha_{k,s_k} \end{vmatrix}. \tag{3.19}$$

*Then, with $\Psi_p(z)$ as in* (3.13), *and with $p > k + \deg(u)$, we have*

(3.20)

$$\begin{aligned}\Delta(z) =&(-1)^k \psi_{1,p}(z) \\ &\times \sum_{1\le s_0<s_1<\cdots<s_k\le\mu} \widehat{T}^{(p)}_{s_0,s_1,\ldots,s_k}(z) V(z_{s_0}, z_{s_1}, \ldots, z_{s_k}) \times \left[\prod_{i=0}^{k} \Psi_p(z_{s_i})\right]^{-1}.\end{aligned}$$

*Proof.* Taking $p > k+\deg(u)$, and invoking Lemma 3.4 in the determinant $\Delta(z)$ of Lemma 3.5, we first have

$$\begin{aligned}\Delta_j(z) &= \psi_{1,p}(z) F[z, \xi_{j+1}, \ldots, \xi_p] = \psi_{1,p}(z) \sum_{s=1}^{\mu} e_s(z) \frac{\psi_{1,j}(z_s)}{\psi_{1,p}(z_s)} \\ &= \psi_{1,p}(z) \sum_{s=1}^{\mu} \widehat{e}^{(p)}_s(z) \frac{\psi_{1,j}(z_s)}{\Psi_p(z_s)},\end{aligned} \tag{3.21}$$

in addition to (3.15). Substituting (3.21) and (3.15) in (3.11), and factoring out $\psi_{1,p}(z)$ from the first row, we thus have

$$\Delta(z) = (-1)^k \psi_{1,p}(z) W(z), \tag{3.22}$$

where

(3.23) $W(z) =$

$$\begin{vmatrix} \sum_{s_0} \widehat{e}^{(p)}_{s_0}(z)\frac{\psi_{1,0}(z_{s_0})}{\Psi_p(z_{s_0})} & \sum_{s_0} \widehat{e}^{(p)}_{s_0}(z)\frac{\psi_{1,1}(z_{s_0})}{\Psi_p(z_{s_0})} & \cdots & \sum_{s_0} \widehat{e}^{(p)}_{s_0}(z)\frac{\psi_{1,k}(z_{s_0})}{\Psi_p(z_{s_0})} \\ \sum_{s_1} \alpha_{1,s_1}\frac{\psi_{1,0}(z_{s_1})}{\Psi_p(z_{s_1})} & \sum_{s_1} \alpha_{1,s_1}\frac{\psi_{1,1}(z_{s_1})}{\Psi_p(z_{s_1})} & \cdots & \sum_{s_1} \alpha_{1,s_1}\frac{\psi_{1,k}(z_{s_1})}{\Psi_p(z_{s_1})} \\ \sum_{s_2} \alpha_{2,s_2}\frac{\psi_{1,0}(z_{s_2})}{\Psi_p(z_{s_2})} & \sum_{s_2} \alpha_{2,s_2}\frac{\psi_{1,1}(z_{s_2})}{\Psi_p(z_{s_2})} & \cdots & \sum_{s_2} \alpha_{2,s_2}\frac{\psi_{1,k}(z_{s_2})}{\Psi_p(z_{s_2})} \\ \vdots & \vdots & & \vdots \\ \sum_{s_k} \alpha_{k,s_k}\frac{\psi_{1,0}(z_{s_k})}{\Psi_p(z_{s_k})} & \sum_{s_k} \alpha_{k,s_k}\frac{\psi_{1,1}(z_{s_k})}{\Psi_p(z_{s_k})} & \cdots & \sum_{s_k} \alpha_{k,s_k}\frac{\psi_{1,k}(z_{s_k})}{\Psi_p(z_{s_k})} \end{vmatrix}.$$

Proceeding as in the proof of Theorem 3.6, we first take the summations outside. Following that, we take out the common factors from each row of the remaining determinant. We obtain

$$W(z) = \sum_{s_0}\sum_{s_1}\cdots\sum_{s_k} \widehat{e}^{(p)}_{s_0}(z)\left(\prod_{i=1}^{k}\alpha_{i,s_i}\right)\left[\prod_{i=0}^{k}\Psi_p(z_{s_i})\right]^{-1} X(z_{s_0}, z_{s_1}, \ldots, z_{s_k}),$$

with $X(y_0, y_1, y_2, \ldots, y_n)$ as given in (3.16). Since the product

$$\left[\prod_{i=0}^{k}\Psi_p(z_{s_i})\right]^{-1} X(z_{s_0}, z_{s_1}, \ldots, z_{s_k})$$

is odd under an interchange of any two of the indices $s_0, s_1, \ldots, s_k$, Lemma 3.1 applies. Finally, invoking also (3.17), we obtain the result in (3.20). ∎

3.3. Algebraic structure of $R_{p,k}(z)$. Finally, combining (3.14) and (3.20) in (3.10), we obtain a simple and elegant expression for $F(z) - R_{p,k}(z)$. This is the subject of the following theorem.

Theorem 3.8: *For the error in $R_{p,k}(z)$, with $p > k + \deg(u)$, we have the closed-form expression*

(3.24) $F(z) - R_{p,k}(z) = \psi_{1,p}(z)$

$$\times \frac{\displaystyle\sum_{1\le s_0<s_1<\cdots<s_k\le\mu} \widehat{T}^{(p)}_{s_0,s_1,\ldots,s_k}(z) V(z_{s_0}, z_{s_1}, \ldots, z_{s_k})\left[\prod_{i=0}^{k}\Psi_p(z_{s_i})\right]^{-1}}{\displaystyle\sum_{1\le s_1<s_2<\cdots<s_k\le\mu} T_{s_1,s_2,\ldots,s_k} V(z, z_{s_1}, z_{s_2}, \ldots, z_{s_k})\left[\prod_{i=1}^{k}\Psi_p(z_{s_i})\right]^{-1}}.$$

*Remark:* When $k=\mu$ in Theorem 3.8, the summation in the numerator on the right-hand side of (3.24) is empty. Thus, this theorem provides an independent proof of the reproducing property of IMMPE.

## 4. Preliminaries for convergence theory

Let $E$ be a closed and bounded set in the $z$-plane, whose complement $K$, including the point at infinity, has a classical Green's function $g(z)$ with a pole at infinity, which is continuous on $\partial E$, the boundary of $E$, and is zero on $\partial E$. For each $\sigma$, let $\Gamma_\sigma$ be the locus $g(z)=\log\sigma$, and let $E_\sigma$ denote the interior of $\Gamma_\sigma$. Then, $E_1$ is the interior of $E$ and, for $1<\sigma<\sigma'$, there holds $E\subset E_\sigma\subset E_{\sigma'}$.

For each $p\in\{1,2,\ldots\}$, let

$$\Xi_p=\left\{\xi_1^{(p)},\xi_2^{(p)},\ldots,\xi_{p+1}^{(p)}\right\} \tag{4.1}$$

be the set of interpolation points used in constructing the IMMPE interpolant $R_{p,k}(z)$. Assume that the sets $\Xi_p$ are such that $\xi_i^{(p)}$ have no limits points in $K$ and

$$\lim_{p\to\infty}\left|\prod_{i=1}^{p+1}\left(z-\xi_i^{(p)}\right)\right|^{1/p}=\kappa\Phi(z);\quad \kappa=\operatorname{cap}(E),\quad \Phi(z)=\exp[g(z)], \tag{4.2}$$

uniformly in $z$ on every compact subset of $K$, where $\operatorname{cap}(E)$ is the logarithmic capacity of $E$ defined by

$$\operatorname{cap}(E)=\lim_{n\to\infty}\Big(\min_{r\in\mathcal{P}_n}\max_{z\in E}|r(z)|\Big)^{1/n};\quad \mathcal{P}_n=\big\{r(z): r\in\Pi_n \text{ and monic}\big\}.$$

Such sequences $\left\{\xi_1^{(p)},\xi_2^{(p)},\ldots,\xi_{p+1}^{(p)}\right\}$, $p=1,2,\ldots$, exist, see Walsh [12, p. 74]. Note that, in terms of $\Phi(z)$, the locus $\Gamma_\sigma$ is defined by $\Phi(z)=\sigma$ for $\sigma>1$, while $\partial E=\Gamma_1$ is simply the locus $\Phi(z)=1$.

Recalling that $\prod_{i=1}^{p+1}\left(z-\xi_i^{(p)}\right)=\Psi_p(z)$ (see (3.13)), we can write (4.2) also as

$$\lim_{p\to\infty}\left|\Psi_p(z)\right|^{1/p}=\kappa\Phi(z), \tag{4.3}$$

uniformly in $z$ on every compact subset of $K$.

It is clear that if $z'\in\Gamma_{\sigma'}$ and $z''\in\Gamma_{\sigma''}$ and $1<\sigma'<\sigma''$, then $\Phi(z')<\Phi(z'')$.

LEMMA 4.1: *Let $K'$ be a compact subset of $K$. Then, for every $\epsilon>0$, there exists an integer $p_0$ depending only on $\epsilon$, such that*

$$\big[(1-\epsilon)\kappa\Phi(z)\big]^p < \big|\Psi_p(z)\big| < \big[(1+\epsilon)\kappa\Phi(z)\big]^p, \quad \text{for all } z\in K' \text{ and for all } p>p_0. \tag{4.4}$$

*Proof.* Since (4.3) holds uniformly in $K'$, for every $\epsilon > 0$, there is an integer $p_0$ independent of $z \in K'$, such that

$$\left|\frac{\big|\Psi_p(z)\big|^{1/p}}{\kappa\Phi(z)} - 1\right| < \epsilon, \quad \text{for every } p > p_0.$$

From this, the result in (4.4) follows. ∎

Lemma 4.2: *For every $\epsilon > 0$, there is an integer $p_0$ depending only on $\epsilon$, such that*

$$\big|\Psi_p(z)\big| < \big[(1+\epsilon)\kappa\big]^p, \quad \text{for all } z\in E \text{ and for all } p>p_0. \tag{4.5}$$

*As a result, we also have that*

$$\limsup_{p\to\infty} \big|\Psi_p(z)\big|^{1/p} \le \kappa \quad \text{for all } z \in E. \tag{4.6}$$

*Proof.* For all $z \in E$ and every $\sigma > 1$, by the maximum modulus theorem, there exists $z^* \in \Gamma_\sigma$, such that

$$\big|\Psi_p(z)\big| \le \big|\Psi_p(z^*)\big|.$$

Next, by the preceding lemma, given $\epsilon > 0$, there exists an integer $p_0$ depending only on $\epsilon$, such that

$$\big|\Psi_p(z^*)\big|^{1/p} < \big[1+\epsilon/(2+\epsilon)\big]\kappa\Phi(z^*) = \big[1+\epsilon/(2+\epsilon)\big]\kappa\sigma.$$

Now, choose $\sigma = 1 + \epsilon/2$, and note that, as $\epsilon \to 0$, the curve $\Gamma_\sigma$ belongs to a fixed compact subset $K'$ of $K$. This results in (4.5), from which (4.6) follows immediately. ∎

Lemma 4.3: *Let* (i) $z', z'' \in K$ *and* $\Phi(z') < \Phi(z'')$, *or* (ii) $z' \in E$ *and* $z'' \in K$. *Then*

$$\lim_{p\to\infty} \left|\frac{\Psi_p(z')}{\Psi_p(z'')}\right|^{1/p} = \frac{\Phi_p(z')}{\Phi_p(z'')} < 1, \quad \textit{case } \text{(i)}. \tag{4.7}$$

$$\limsup_{p\to\infty} \left|\frac{\Psi_p(z')}{\Psi_p(z'')}\right|^{1/p} \le \frac{1}{\Phi_p(z'')} < 1, \quad \textit{case } \text{(ii)}. \tag{4.8}$$

*In both cases,*

$$\lim_{p\to\infty} \frac{\Psi_p(z')}{\Psi_p(z'')} = 0. \tag{4.9}$$

*Proof.* For $\epsilon > 0$ arbitrary, it follows from Lemma 4.1 that, when $z', z'' \in K$, there exists an integer $p_0$, such that

$$\frac{|\Psi_p(z')|}{|\Psi_p(z'')|} < \left[\frac{(1+\epsilon)\Phi(z')}{(1-\epsilon)\Phi(z'')}\right]^p, \quad \text{for all } p > p_0.$$

Now, by the assumption that $\Phi(z') < \Phi(z'')$, it is clear that we can choose $\epsilon$ small enough to ensure

$$\frac{(1+\epsilon)\Phi(z')}{(1-\epsilon)\Phi(z'')} < 1.$$

The result of case (i) now follows. The result of case (ii) follows by invoking Lemma 4.2 and proceeding similarly. ∎

## 5. Convergence theory for vector-valued rational $F(z)$ with simple poles

In this section, we provide a convergence theory, in case $F(z)$ is a vector-valued rational function with simple poles as in (3.1), for the sequences $\{R_{p,k}(z)\}_{p=1}^{\infty}$ with $k < \mu$ and fixed. (Note that by the reproducing property mentioned in Section 1, for $k = \mu$, $R_{p,k}(z) = F(z)$ for all $p \geq p_0$, where $p_0 - 1$ is the degree of the numerator of $F(z)$.) Also, as we will let $p \to \infty$ in our analysis, the condition that $p > k + \deg(u)$ is satisfied for all large $p$.

We continue to use the notation of the preceding sections. We now turn to $F(z)$ in (3.1). We assume that $F(z)$ is analytic in $E$. This implies that its poles $z_1, \ldots, z_\mu$ are all in $K$. Now we order the poles of $F(z)$ such that

$$\Phi(z_1) \leq \Phi(z_2) \leq \cdots \leq \Phi(z_\mu). \tag{5.1}$$

By Lemma 4.3, if $z'$ and $z''$ are two different poles of $F(z)$, and $\Phi(z') < \Phi(z'')$, then $z'$ and $z''$ lie on two different loci $\Gamma_{\sigma'}$ and $\Gamma_{\sigma''}$. In addition, $\sigma' < \sigma''$, that is, the set $E_{\sigma'}$ is in the interior of $E_{\sigma''}$.

### 5.1. Convergence analysis for $V_{p,k}(z)$.

We now state a Koenig-type convergence theorem for $V_{p,k}(z)(z)$ and another theorem concerning its zeros. These results are analogous to, and in the spirit of, the ones given in Sidi [5] for denominators of Padé approximants. We remind the reader that the $q_i$ are as in (2.9), and the $v_j$ are as in (3.1).

Theorem 5.1: *Assume*

$$\Phi(z_k) < \Phi(z_{k+1}) = \cdots = \Phi(z_{k+r}) < \Phi(z_{k+r+1}), \tag{5.2}$$

*in addition to* (5.1). *In case* $k + r = \mu$, *we define* $\Phi(z_{k+r+1}) = \infty$. *Assume also that*

$$T_{1,\dots,k} = \begin{vmatrix} (q_1, v_1) & \cdots & (q_1, v_k) \\ \vdots & & \vdots \\ (q_k, v_1) & \cdots & (q_k, v_k) \end{vmatrix} \neq 0. \tag{5.3}$$

*Then, there holds*

$$Q(z) = (-1)^k T_{1,\dots,k} V(z, z_1, \dots, z_k) \left[ \prod_{i=1}^{k} \Psi_p(z_i) \right]^{-1} \left[ 1 + O\left( \frac{\Psi_p(z_k)}{\widetilde{\Psi}_{p,k}} \right) \right] \quad \text{as } p \to \infty, \tag{5.4}$$

*uniformly in every compact subset of* $\mathbb{C} \setminus \{z_1, z_2, \dots, z_k\}$, *where*

$$\left|\widetilde{\Psi}_{p,k}\right| = \min_{1 \le j \le r} \left|\Psi_p(z_{k+j})\right|. \tag{5.5}$$

*Thus, with the normalization that* $c_k = 1$, *and letting*

$$S(z) = \prod_{i=1}^{k} (z - z_i), \tag{5.6}$$

*there holds*

$$V_{p,k}(z) - S(z) = O\left( \frac{\Psi_p(z_k)}{\widetilde{\Psi}_{p,k}} \right) \quad \text{as } p \to \infty, \tag{5.7}$$

*from which we also have*

$$\limsup_{p \to \infty} \left| V_{p,k}(z) - S(z) \right|^{1/p} \le \frac{\Phi(z_k)}{\Phi(z_{k+1})}. \tag{5.8}$$

*Proof.* By (5.1) and (5.2) and Lemma 4.3, the largest term in (3.14) as $p \to \infty$ is that with the indices $(s_1, \dots, s_k) = (1, \dots, k)$. The next largest terms are those with $(s_1, \dots, s_k) = (1, \dots, k-1, k+j)$, $1 \le j \le r$. Obviously, we have $\lim_{p \to \infty} \left[\Psi_p(z_k)/\widetilde{\Psi}_{p,k}\right] = 0$. This completes the proof of (5.4). The proof of (5.7) can be achieved by noting that

$$V(z, z_1, \dots, z_k) = (-1)^k V(z_1, \dots, z_k) \prod_{i=1}^{k} (z - z_i). \tag{5.9}$$

The proof of (5.8) follows from (5.7) and (4.3). ∎

Theorem 5.1 implies that $V_{p,k}(z)$ has precisely $k$ zeros that tend to those of $S(z)$. Let us denote the zeros of $V_{p,k}(z)$ by $z_m^{(p)}$, $m = 1, \ldots, k$. Then $\lim_{p\to\infty} z_m^{(p)} = z_m$, $m = 1, \ldots, k$. In the next theorem, we provide the rate of convergence of each of these zeros.

THEOREM 5.2: *Under the conditions of Theorem 5.1, there holds*

$$z_m^{(p)} - z_m = O\left(\frac{\Psi_p(z_m)}{\widetilde{\Psi}_{p,k}}\right) \quad \text{as } p \to \infty, \tag{5.10}$$

*with $\widetilde{\Psi}_{p,k}$ as in (5.5). From this, it follows that*

$$\limsup_{p\to\infty} \left|z_m^{(p)} - z_m\right|^{1/p} \le \frac{\Phi(z_m)}{\Phi(z_{k+1})}, \quad m = 1, \ldots, k. \tag{5.11}$$

*In case $r = 1$ in (5.2), that is,*

$$\Phi(z_k) < \Phi(z_{k+1}) < \Phi(z_{k+2}), \tag{5.12}$$

*and assuming that $T_{1,\ldots,m-1,m+1,\ldots,k+1} \neq 0$, we have the more refined result*

$$z_m^{(p)} - z_m \sim C_m \frac{\Psi_p(z_m)}{\Psi_p(z_{k+1})} \quad \text{as } p \to \infty, \tag{5.13}$$

$$C_m = (-1)^{k-m} \frac{T_{1,\ldots,m-1,m+1,\ldots,k+1}}{T_{1,\ldots,k}} (z_{k+1} - z_m) \prod_{\substack{i=1\\ i\neq m}}^{k} \frac{z_{k+1} - z_i}{z_m - z_i}.$$

*Proof.* First, we have

$$0 = V_{p,k}(z_m^{(p)}) = V_{p,k}(z_m) + \int_{z_m}^{z_m^{(p)}} V'_{p,k}(t)\, dt,$$

where the integral is over the directed line segment from $z_m$ to $z_m^{(p)}$. Hence,

$$0 = V_{p,k}(z_m) + V'_{p,k}(z_m)(z_m^{(p)} - z_m) + e_m^{(p)}; \quad e_m^{(p)} = \int_{z_m}^{z_m^{(p)}} \left[V'_{p,k}(t) - V'_{p,k}(z_m)\right] dt.$$

Because $\lim_{p\to\infty} z_m^{(p)} = z_m$, and $\{V_{p,k}(z)\}_{p=0}^{\infty}$ is a uniformly convergent sequence of polynomials of degree $k$, we have that $e_m^{(p)} = O\big(|z_m^{(p)} - z_m|^2\big)$ as $p \to \infty$. Next,

$$z_m^{(p)} - z_m = -\frac{V_{p,k}(z_m)}{V'_{p,k}(z_m) + e_m^{(p)}/(z_m^{(p)} - z_m)}.$$

Finally, because

$$\lim_{p\to\infty} z_m^{(p)} = z_m, \quad \lim_{p\to\infty} V'_{p,k}(z_m) \neq 0, \quad \text{and} \quad \lim_{p\to\infty} e_m^{(p)}/(z_m^{(p)} - z_m) = 0,$$

we also have the asymptotic equality

$$z_m^{(p)} - z_m \sim -\frac{V_{p,k}(z_m)}{V'_{p,k}(z_m)} \quad \text{as } p \to \infty. \tag{5.14}$$

Since $Q(z)$ in (2.11) is a constant multiple of $V_{p,k}(z)$, this asymptotic equality can be rewritten as in

$$z_m^{(p)} - z_m \sim -\frac{Q(z_m)}{Q'(z_m)} \quad \text{as } p \to \infty. \tag{5.15}$$

By (3.14),

$$Q'(z_m) = (-1)^k \sum_{1\le s_1<s_2<\cdots<s_k\le\mu} T_{s_1,\ldots,s_k} a^{(m)}_{s_1,\ldots,s_k} \left[\prod_{i=1}^{k} \Psi_p(z_{s_i})\right]^{-1}, \tag{5.16}$$

$$a^{(m)}_{s_1,s_2,\ldots,s_k} = \frac{d}{dz} V(z, z_{s_1}, z_{s_2}, \ldots, z_{s_k})\big|_{z=z_m}.$$

Proceeding as in the proof of Theorem 5.1, we see that, because

$$a^{(m)}_{1,\ldots,k} = (-1)^k V(z_1,\ldots,z_k) \prod_{\substack{i=1\\ i\neq m}}^{k} (z_m - z_i) \neq 0,$$

the dominant term as $p \to \infty$ in the summation of (5.16) is that with $(s_1,\ldots,s_k) = (1,\ldots,k)$, the rest of the terms being negligible. Therefore, $Q'(z_m)$ satisfies the asymptotic equality

$$Q'(z_m) \sim (-1)^k T_{1,\ldots,k} a^{(m)}_{1,\ldots,k} \left[\prod_{i=1}^{k} \Psi_p(z_i)\right]^{-1} \quad \text{as } p \to \infty. \tag{5.17}$$

Setting $z = z_m$ in (3.14), and recalling that $V(y_0, y_1, \ldots, y_k)$ vanishes when any two of the $y_j$ are equal, we have

$$Q(z_m) = (-1)^k \sum_{\substack{1\le s_1<\cdots<s_k\le\mu\\ s_1,\ldots,s_k\neq m}} T_{s_1,\ldots,s_k} V(z_m, z_{s_1}, z_{s_2}, \ldots, z_{s_k}) \left[\prod_{i=1}^{k} \Psi_p(z_{s_i})\right]^{-1}. \tag{5.18}$$

The dominant terms in this summation are those with

$$(s_1,\ldots,s_k) = (1,\ldots,m-1,m+1,\ldots,k,k+j), \quad 1\le j\le r,$$

the rest of the terms being negligible. Thus,

$$Q(z_m) = O\left(\left[\prod_{i=1}^{k} \Psi_p(z_i)\right]^{-1} \frac{\Psi_p(z_m)}{\widetilde{\Psi}_{p,k}}\right) \quad \text{as } p \to \infty. \tag{5.19}$$

Combining (5.17) and (5.19) in (5.15), we obtain (5.10). The result in (5.11) follows from (5.10) and (4.3).

In case $r = 1$, taking only the term with

$$(s_1, \dots, s_k) = (1, \dots, m-1, m+1, \dots, k, k+1),$$

in (5.18), we have the asymptotic equality

$$\begin{aligned} Q(z_m) \sim (-1)^k T_{1,\dots,m-1,m+1,\dots,k+1} V(z_m, z_1, \dots, z_{m-1}, z_{m+1}, \dots, z_{k+1}) \\ \times \left[\prod_{i=1}^{k} \Psi_p(z_i)\right]^{-1} \frac{\Psi_p(z_m)}{\Psi_p(z_{k+1})} \quad \text{as } p \to \infty. \end{aligned} \tag{5.20}$$

The result in (5.13) is now obtained by combining (5.17) and (5.20) in (5.15), and by invoking

$$\begin{aligned} &V(z_m, z_1, \dots, z_{m-1}, z_{m+1}, \dots, z_{k+1}) \\ &\qquad = (-1)^{m-1} V(z_1, \dots, z_{k+1}). \\ &\qquad = (-1)^{m-1} V(z_1, \dots, z_k) \prod_{i=1}^{k} (z_{k+1} - z_i). \qquad \blacksquare \end{aligned}$$

5.2. Convergence analysis for $R_{p,k}(z)$. We now continue to the analysis of $F(z) - R_{p,k}(z)$, as $p \to \infty$. Throughout the rest of this work, $\|Y\|$ denotes the vector norm of $Y \in \mathbb{C}^N$.

Theorem 5.3: *Under the conditions of Theorem 5.1, $R_{p,k}(z)$ exists and is unique and satisfies*

$$F(z) - R_{p,k}(z) = O\left(\frac{\Psi_p(z)}{\widetilde{\Psi}_{p,k}}\right) \quad \text{as } p \to \infty, \tag{5.21}$$

*uniformly on every compact subset of $\mathbb{C} \setminus \{z_1, \dots, z_\mu\}$, with $\widetilde{\Psi}_{p,k}$ as defined in* (5.5). *From this, it also follows that*

$$\limsup_{p\to\infty} \left\|F(z) - R_{p,k}(z)\right\|^{1/p} \le \frac{\Phi(z)}{\Phi(z_{k+1})}, \quad z \in \widetilde{K} = K \setminus \{z_1, \dots, z_\mu\}, \tag{5.22}$$

*uniformly on each compact subset of $\widetilde{K}$, and*

$$\limsup_{p\to\infty} \left\|F(z) - R_{p,k}(z)\right\|^{1/p} \le \frac{1}{\Phi(z_{k+1})}, \quad z \in E, \tag{5.23}$$

*uniformly on $E$. Thus, uniform convergence takes place for $z$ in any compact subset of the set $\widetilde{K}_k$, where*

$$\widetilde{K}_k = \operatorname{int}\Gamma_{\sigma_k} \setminus \{z_1, \ldots, z_k\}; \quad \sigma_k = \Phi(z_{k+1}).$$

*When $r = 1$ in (5.2), that is, when*

$$\Phi(z_k) < \Phi(z_{k+1}) < \Phi(z_{k+2}), \tag{5.24}$$

*and $\widehat{T}^{(p)}_{1,\ldots,k+1}(z) \neq 0$ in addition to (5.3), we have the more refined result*

$$F(z) - R_{p,k}(z) \sim B_p(z)\frac{\psi_{1,p}(z)}{\Psi_p(z_{k+1})} \quad \text{as } p \to \infty, \tag{5.25}$$

$$B_p(z) = (-1)^k \frac{\widehat{T}^{(p)}_{1,\ldots,k+1}(z)}{T_{1,\ldots,k}} \prod_{i=1}^{k} \frac{z_{k+1} - z_i}{z - z_i},$$

*and $B_p(z)$ is bounded for all large $p$.*

*Proof.* We have already analyzed $Q(z)$ in Theorem 5.1 and obtained the result in (5.4), from which we also have the asymptotic equality

$$Q(z) \sim (-1)^k T_{1,\ldots,k} V(z, z_1, \ldots, z_k)\left[\prod_{i=1}^{k} \Psi_p(z_i)\right]^{-1} \quad \text{as } p \to \infty \tag{5.26}$$

that holds uniformly in every compact subset of $\mathbb{C} \setminus \{z_1, \ldots, z_\mu\}$. This shows that, for all large $p$, $V_{p,k}(z)$ is such that $V_{p,k}(\xi_i) \neq 0$, for $i = 1, \ldots, p$, and that the condition in (2.13) is satisfied because

$$\begin{vmatrix} u_{1,0} & u_{1,1} & \cdots & u_{1,k-1} \\ u_{2,0} & u_{2,1} & \cdots & u_{2,k-1} \\ \vdots & \vdots & & \vdots \\ u_{k,0} & u_{k,1} & \cdots & u_{k,k-1} \end{vmatrix} = (-1)^k Q^{(k)}(z)/k!,$$

and that, by (5.26),

$$Q^{(k)}(z) \sim k! T_{1,\ldots,k} V(z_1, \ldots, z_k)\left[\prod_{i=1}^{k} \Psi_p(z_i)\right]^{-1} \neq 0 \quad \text{as } p \to \infty.$$

Under these, $R_{p,k}(z)$ exists and is unique, as mentioned in Section 2.

To complete the proof, we need to analyze the asymptotic behavior of $\Delta(z)$. From (3.20) in Theorem 3.7, arguing as before, we have that the dominant terms in the summation in (3.20) are those having indices

$$(s_0, s_1, \ldots, s_k) = (1, \ldots, k, k+j), \quad 1 \leq j \leq r.$$

The rest of the terms are negligible by Lemma 4.3. Thus,

$$\Delta(z) = O\left(\left[\prod_{i=1}^{k} \Psi_p(z_i)\right]^{-1} \frac{\psi_{1,p}(z)}{\widetilde{\Psi}_{p,k}}\right) \quad \text{as } p \to \infty. \tag{5.27}$$

This also holds uniformly in every compact subset of $K \setminus \{z_1, \ldots, z_\mu\}$ because, on account of the fact that

$$\widehat{e}_s^{(p)}(z) = v_s(z_s - \xi_{p+1}^{(p)})/(z - z_s),$$

$\widehat{T}^{(p)}_{s_0,s_1,\ldots,s_k}(z)$ are analytic in these subsets and are also bounded for all large $p$ since the $\xi_i^{(p)}$ are bounded for all large $p$. Combining (5.26) and (5.27) in (3.10), we obtain (5.21). The result in (5.22) follows from (5.21).

In case $r = 1$, there is only one dominant term in the summation of (3.20), namely, the one with $(s_0, s_1, \ldots, s_k) = (1, \ldots, k+1)$. Thus, $\Delta(z)$ satisfies the asymptotic equality

$$\Delta(z) \sim (-1)^k \psi_{1,p}(z) \widehat{T}^{(p)}_{1,\ldots,k+1}(z) V(z_1, \ldots, z_{k+1}) \left[\prod_{i=1}^{k+1} \Psi_p(z_i)\right]^{-1} \quad \text{as } p \to \infty. \tag{5.28}$$

Combining (5.26) and (5.28) in (3.10), we obtain (5.25). ∎

## 6. Convergence theory for meromorphic $F(z)$ with simple poles

Let the sets of interpolation points $\{\xi_1^{(p)}, \ldots, \xi_{p+1}^{(p)}\}$ be as in the preceding section. We now turn to the convergence analysis of $R_{p,k}(z)$ as $p \to \infty$, when the function $F(z)$ is analytic in $E$ and meromorphic in $E_\rho = \operatorname{int} \Gamma_\rho$, where $\Gamma_\rho$, as before, is the locus $\Phi(z) = \rho$ for some $\rho > 1$. Assume that $F(z)$ has $\mu$ simple poles $z_1, \ldots, z_\mu$ in $E_\rho$. Thus, $F(z)$ has the following form:

$$F(z) = \sum_{s=1}^{\mu} \frac{v_s}{z - z_s} + \Theta(z), \tag{6.1}$$

$\Theta(z)$ being analytic in $E_\rho$.

The treatment of this case is based entirely on that of the preceding section, the differences being minor. Note that the polynomial $u(z)$ of (3.1) is now replaced by $\Theta(z)$ in (6.1). Previously, we had $u[\xi_m,\ldots,\xi_n]=0$ for all large $n-m$, as a consequence of which, we had (3.15) for $u_{i,j}$ and (3.21) for $\Delta_j(z)$. Instead of these, we now have

$$u_{i,j}=-\sum_{s=1}^{\mu}\alpha_{i,s}\frac{\psi_{1,j}(z_s)}{\Psi_p(z_s)}+\big(q_i,\Theta[\xi_{j+1},\ldots,\xi_{p+1}]\big), \tag{6.2}$$

with $\alpha_{i,s}$ as in (3.8), and

$$\Delta_j(z)=\psi_{1,p}(z)\Bigg(\sum_{s=1}^{\mu}\widehat{e}_s^{(p)}(z)\frac{\psi_{1,j}(z_s)}{\Psi_p(z_s)}+\Theta[z,\xi_{j+1},\ldots,\xi_p]\Bigg), \tag{6.3}$$

with $\widehat{e}_s^{(p)}(z)$ as in (3.18).

It is clear that the treatment of the general meromorphic $F(z)$ will be the same as that of the rational $F(z)$ provided the contributions from $\Theta(z)$ to $u_{i,j}$ and $\Delta_j(z)$, as $p\to\infty$, are negligible compared to the rest of the terms in (6.2) and (6.3). We explore this point next.

Lemma 6.1: *With $F(z)$ as in the first paragraph, there holds*

$$\limsup_{p\to\infty}\big\|\Theta[\xi_{j+1}^{(p)},\ldots,\xi_{p+1}^{(p)}]\big\|^{1/p}\le 1/(\kappa\rho). \tag{6.4}$$

*There also holds*

$$\limsup_{p\to\infty}\big\|\Theta[z,\xi_{j+1}^{(p)},\ldots,\xi_{p}^{(p)}]\big\|^{1/p}\le 1/(\kappa\rho), \tag{6.5}$$

*uniformly in every compact subset of $E_\rho$.*

*Proof.* Let $\rho_1,\rho_2$ be arbitrary numbers satisfying $1<\rho_2<\rho_1<\rho$. For $i=1,2$, denote by $C_i$ the locus $\Gamma_{\rho_i}$ (i.e., $\Phi(z)=\rho_i$), and let $S_i$ be the closure of the interior of $C_i$. Thus, the closed curves $C_1,C_2,\Gamma_\rho$ have no common points, and $S_2\subset S_1\subset S$. Clearly, $\Theta(z)$ is analytic in $S_1$ and $S_2$. Since the $\xi_i^{(p)}$ do not have a limit point in $K$, from Hermite's formula, we have that

$$\Theta[\xi_{j+1}^{(p)},\ldots,\xi_{p+1}^{(p)}]=\frac{1}{2\pi\mathrm{i}}\oint_{C_1}\frac{\Theta(\zeta)}{\prod_{i=j+1}^{p+1}(\zeta-\xi_i^{(p)})}\,d\zeta=\frac{1}{2\pi\mathrm{i}}\oint_{C_1}\frac{\psi_{1,j}(\zeta)\,\Theta(\zeta)}{\Psi_p(\zeta)}\,d\zeta$$

for all large $p$. Therefore,

$$\big\|\Theta[\xi_{j+1}^{(p)},\ldots,\xi_{p+1}^{(p)}]\big\|\le\frac{1}{2\pi}\oint_{C_1}\frac{\|\psi_{1,j}(\zeta)\,\Theta(\zeta)\|}{|\Psi_p(\zeta)|}\,|d\zeta|.$$

Here, we have used the fact that (see, for example, Ortega [3, pp. 142–143])

$$\left\| \int_{C_1} H(\zeta)\, d\zeta \right\| \leq \int_{C_1} \|H(\zeta)\|\, |d\zeta|$$

when $H(\zeta)$ is a vector-valued function continuous on $C_1$. Now, from Lemma 4.1, we know that given $\epsilon > 0$, there exists an integer $p_0$ independent of $\zeta$ such that, for all $p > p_0$, there holds $\left|\Psi_p(\zeta)\right| > [(1-\epsilon)\kappa\Phi(\zeta)]^p$. In addition, $\Phi(\zeta) = \rho_1$ on $C_1$. Therefore,

$$\left\|\Theta[\xi^{(p)}_{j+1}, \ldots, \xi^{(p)}_{p+1}]\right\| \leq \frac{L_1}{2\pi} \frac{A^{(p)}_j}{[(1-\epsilon)\kappa\rho_1]^p}, \quad A^{(p)}_j \equiv \max_{\zeta\in C_1} \|\psi_{1,j}(\zeta)\, \Theta(\zeta)\|,$$

where $L_1$ is the length of $C_1$. Now, because $j \leq k$ and $k$ is fixed, $A^{(p)}_j$ are bounded in $p$. As a result,

$$\limsup_{p\to\infty} \left\|\Theta[\xi^{(p)}_{j+1}, \ldots, \xi^{(p)}_{p+1}]\right\|^{1/p} \leq \frac{1}{\kappa\rho_1(1-\epsilon)}.$$

Since $\epsilon > 0$ and $\rho_1 < \rho$ are arbitrary, the result in (6.4) now follows.

To prove (6.5), we proceed similarly. Taking $z \in S_2$, Hermite's formula now reads

$$\begin{aligned}\Theta[z, \xi^{(p)}_{j+1}, \ldots, \xi^{(p)}_{p}] &= \frac{1}{2\pi \mathrm{i}} \oint_{C_1} \frac{\Theta(\zeta)}{\prod_{i=j+1}^{p} (\zeta - \xi^{(p)}_i)} \frac{d\zeta}{\zeta - z} \\ &= \frac{1}{2\pi \mathrm{i}} \oint_{C_1} \frac{(\zeta - \xi^{(p)}_{p+1})\, \psi_{1,j}(\zeta)\, \Theta(\zeta)}{(\zeta - z)\, \Psi_p(\zeta)}\, d\zeta\end{aligned}$$

for all large $p$. Proceeding exactly as before, and using the fact that

$$d = \min_{z\in S_2,\, \zeta\in C_1} |\zeta - z| > 0,$$

we obtain

$$\begin{aligned}\left\|\Theta[z, \xi^{(p)}_{j+1}, \ldots, \xi^{(p)}_{p}]\right\| &\leq \frac{L_1}{2\pi d} \frac{B^{(p)}_j}{[(1-\epsilon)\kappa\rho_1]^p}, \\ B^{(p)}_j &\equiv \max_{\zeta\in C_1} \left\|(\zeta - \xi^{(p)}_{p+1})\, \psi_{1,j}(\zeta)\, \Theta(\zeta)\right\|.\end{aligned}$$

The rest is as before. ∎

What we have shown in Lemma 6.1 is that the term $\left(q_i, \Theta[\xi^{(p)}_{j+1}, \ldots, \xi^{(p)}_{p+1}]\right)$ in (6.2) and the term $\psi_{1,p}(z)\Theta[z, \xi^{(p)}_{j+1}, \ldots, \xi^{(p)}_{p}]$ in (6.3) are indeed asymptotically smaller than the rest of the terms. With this information, we can now prove the following theorems for general meromorphic $F(z)$. We recall that the poles $z_1, \ldots, z_\mu$ of $F(z)$ are ordered such that

$$\Phi(z_1) \leq \Phi(z_2) \leq \cdots \leq \Phi(z_\mu) \leq \rho. \tag{6.6}$$

We also adopt the notation of Theorems 5.1, 5.2, and 5.3.

THEOREM 6.2: (i) *When $k < \mu$, assume that*

$$\Phi(z_k) < \Phi(z_{k+1}) = \cdots = \Phi(z_{k+r}) < \begin{cases} \Phi(z_{k+r+1}) & \text{if } k+r < \mu, \\ \rho & \text{if } k+r = \mu, \end{cases} \tag{6.7}$$

*in addition to* (6.6). *Assume also that*

$$T_{1,\ldots,k} = \begin{vmatrix} (q_1, v_1) & \cdots & (q_1, v_k) \\ \vdots & & \vdots \\ (q_k, v_1) & \cdots & (q_k, v_k) \end{vmatrix} \neq 0. \tag{6.8}$$

*Then, all the results of Theorem 5.1 hold.*

(ii) *When $k = \mu$,*

$$\limsup_{p\to\infty} \left| V_{p,k}(z) - S(z) \right|^{1/p} \leq \Phi(z_k)/\rho. \tag{6.9}$$

*uniformly on every compact subset of $\mathbb{C} \setminus \{z_1, \ldots, z_\mu\}$.*

Theorem 6.2 implies that $V_{p,k}(z)$ has precisely $k$ zeros that tend to those of $S(z)$. Let us denote the zeros of $V_{p,k}(z)$ by $z_m^{(p)}$, $m = 1, \ldots, k$. Then $\lim_{p\to\infty} z_m^{(p)} = z_m$, $m = 1, \ldots, k$. In the next theorem, we provide the rate of convergence of each of these zeros.

THEOREM 6.3: *Assume the conditions of Theorem 5.2.*

(i) *When $k < \mu$, all the results of Theorem 5.2 hold.*

(ii) *When $k = \mu$,*

$$\limsup_{p\to\infty} \left| z_m^{(p)} - z_m \right|^{1/p} \leq \Phi(z_m)/\rho, \quad m = 1, \ldots, k. \tag{6.10}$$

THEOREM 6.4: *Assume the conditions of Theorem 5.3. Then $R_{p,k}(z)$ exists and is unique.*

(i) *When $k < \mu$, all the results of Theorem 5.3 hold with*

$$\widetilde{K} = E_\rho \setminus \{z_1, \ldots, z_\mu\}.$$

(ii) *When $k = \mu$, there holds*

$$\limsup_{p\to\infty} \left\|F(z) - R_{p,k}(z)\right\|^{1/p} \leq \Phi(z)/\rho, \quad z \in \widetilde{K} = E_\rho \setminus \{z_1, \ldots, z_\mu\}, \tag{6.11}$$

*uniformly on each compact subset of $\widetilde{K}$, and*

$$\limsup_{p\to\infty} \left\|F(z) - R_{p,k}(z)\right\|^{1/p} \leq 1/\rho, \quad z \in E, \tag{6.12}$$

*uniformly on $E$.*

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